RADIATION OF DIFFUSE ISOTHERMAL CAVITIES

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General equations are derived for the emissivity distribution over the surface, the angular characteristic of the emissivity, and the hemispherical emissivity of diffuse isothermal cavities.

Uniformly heated diffuse cavities are widely used as radiation sources approximating absolute blackbodies. Accordingly, there is interest in such characteristics of the cavities as the emissivity distribution over the surface, the angular characteristic of the emissivity, the hemispherical emissivity, etc. Below we derive general equations for engineering calculations of these characteristics, and we give equations for the surface emissivity of several widely used cavities.

To calculate the distribution of the emissivity over the surface of the cavity we use the procedure used in [1] for the case of the hemispherical emissivity. We assume that a radiative flux $F(\sigma)$ is incident on the cavity at some point σ. Then the flux emitted from the cavity as a result of multiple reflections can be written as

$$F_{\text{emi}}(\sigma) = F(\sigma)\,[\rho u_1(\sigma) + \rho^2 u_2(\sigma) + \ldots + \rho^n u_n(\sigma) + \ldots]. \tag{1}$$

Since there is no absorption in the cavity in the case $\rho = 1$, all the radiation entering the cavity must exit through its aperture; i.e., we would have $F_{\text{emi}}(\sigma) = F(\sigma)$. Accordingly, the series

$$u_1(\sigma) + u_2(\sigma) + \ldots + u_n(\sigma) + \ldots \tag{2}$$

converges and has the finite sum

$$\sum_{n=1}^{\infty} u_n(\sigma) = 1. \tag{3}$$

To simplify the problem we assume that series (2) converges according to a geometric progression, i.e., that for all n we can write

$$u_{n+1}(\sigma)/u_n(\sigma) = k = \text{const} \tag{4}$$

and

$$\sum_{n=1}^{\infty} u_n(\sigma) = u_1(\sigma)/(1-k). \tag{5}$$

This simplification does not introduce an error unacceptable for engineering calculations. From (3) and (5) we find

$$k = 1 - u_1(\sigma). \tag{6}$$

According to our assumption, the series in (1) also turns out to be a geometric progression. Replacing $u_1(\sigma)$ by u_σ, we write the emissivity of point σ of the cavity according to Kirchhoff's law:

$$\varepsilon_\sigma = 1 - \frac{F_{\text{emi}}(\sigma)}{F(\sigma)} = \frac{1-\rho}{1-\rho(1-u_\sigma)} = \frac{\varepsilon}{\varepsilon + (1-\varepsilon)\,u_\sigma}. \tag{7}$$

Using the concept of vector solid angle, which is common in photometry [2], we can write the angular coefficient u_σ as (Fig. 1)

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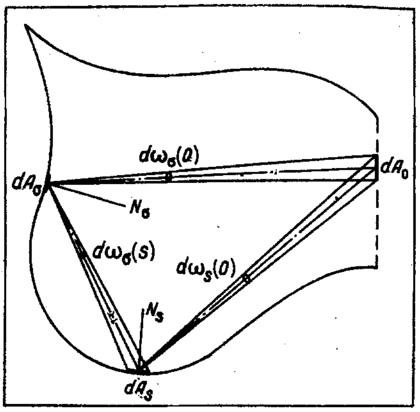

Fig. 1. Cavity of arbitrary shape.

$$u_\sigma = \int\limits_{(0)} \frac{\cos\theta d\omega_\sigma(0)}{\pi} = \frac{\Omega_\sigma(0)}{\pi}. \tag{8}$$

Treating Eq. (7) as a first approximation in solving the integral equation for the emissivity of an arbitrary cavity,

$$\varepsilon_\sigma = \varepsilon + (1-\varepsilon)\int\limits_{(s)} \varepsilon_s \frac{d\Omega_\sigma(s)}{\pi}, \tag{9}$$

we find in the second approximation

$$\varepsilon_\sigma = \frac{\varepsilon}{\varepsilon + (1-\varepsilon)u_\sigma} + (1-\varepsilon)^2\left[\frac{u_\sigma(1-u_\sigma)}{\varepsilon + (1-\varepsilon)u_\sigma} - \frac{1}{\pi}\int\limits_{(s)} \frac{u_s d\Omega_\sigma(s)}{\varepsilon + (1-\varepsilon)u_s}\right]. \tag{10}$$

Equation (10) is more accurate than Eq. (7).

Knowing the distribution of the emissivity over the cavity surface we can calculate the emission characteristic of the cavity. For this purpose we must average the emissivity over those parts of the surface which are observed through the cavity aperture from a given direction:

$$\varepsilon(\varphi, \chi) = \frac{\iint\limits_{(0)} \varepsilon_\sigma(x, y, z)\, d\xi d\eta}{\iint\limits_{(C)} d\xi d\eta}. \tag{11}$$

The coordinates of the point $\sigma(x, y, z)$ onto which a point having coordinates (ξ, η) in the plane of the aperture is projected from this direction are found by solving the equations (Fig. 2)

$$\begin{cases} \Phi(x, y, z) = 0, \\ (x-\xi)^2 + (y-\eta)^2 = z^2 \operatorname{tg}^2\chi, \\ y - \eta = (x - \xi)\operatorname{tg}\varphi, \end{cases} \tag{12}$$

where the first function, $\Phi(x, y, z) = 0$, describes the surface of the cavity; the second equation describes a cone whose vertex is at the point (ξ, η), whose axis is parallel to the 0z axis, and whose vertex half-angle is χ; and the third equation describes a plane passing through the point (ξ, η) at an angle φ with respect to the x0z plane.

After we determine the characteristic $\varepsilon(\varphi, \chi)$ we can integrate it over solid angle to calculate the hemispherical emissivity ε_h. However, it is simpler to calculate ε_h by using the general equation [3]

$$\varepsilon_h = \frac{\varepsilon}{A_0(1-\varepsilon)}\int\limits_{(s)} (1-\varepsilon_\sigma)\, dA_\sigma. \tag{13}$$

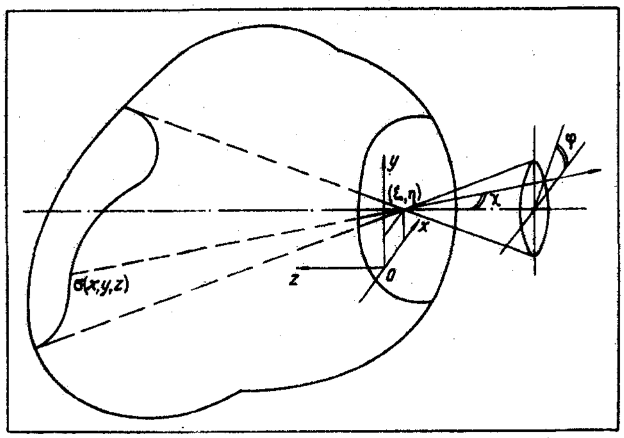

Fig. 2. Diagram illustrating the notation of the coordinates on a cavity of arbitrary shape.

TABLE 1. Emissivity of Diffuse Cylindrical Cavities for the Case $\rho = 0.5$

L	$\varepsilon_{r=0}$				$\varepsilon_{r=1}$				$\bar{\varepsilon}_r$				ε_h			
	[3]	[4]	(7)	(10)	[3]	[4]	(7)	(10)	[5]	[6]	(7)	(10)	[3]	[4]	[1]	(14)
0,2	—	0,514	0,510	0,513	—	0,694	0,690	0,695	—	—	0,550	0,562	—	—	—	—
0,5	0,5693	575	556	571	0,7317	731	725	736	—	0,638	621	641	0,6569	0,625	0,648	0,654
1	6878	700	667	688	7914	789	784	799	0,766	742	724	741	7424	700	736	739
2	8394	843	833	841	8776	866	872	886	884	871	854	863	8084	786	792	800
4	9460	928	944	951	9540	931	950	959	959	969	947	955	8331	864	806	820
6	9768	954	974	980	9793	955	975	981	—	992	974	981	8359	900	807	822
8	9880	966	985	990	9887	969	985	990	989	998	985	990	8367	921	807	823
10	—	974	990	994	—	974	990	994	—	—	990	994	—	—	—	—
∞	1,0000	1,000	1,000	1,000	1,0000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	—	1,000	0,808	0,824

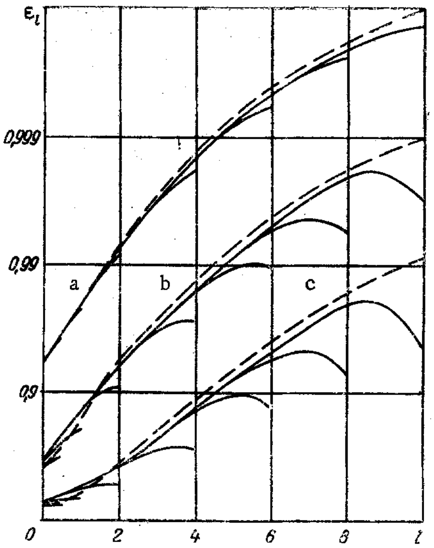

Fig. 3

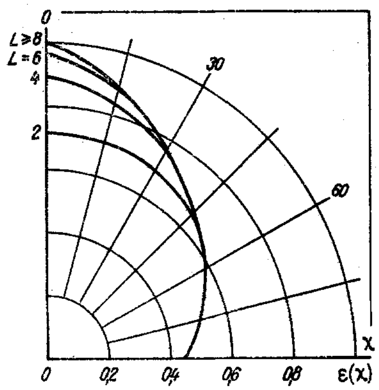

Fig. 4

Fig. 3. Emissivity of the lateral surface of cylindrical cavities of various depths L as a function of l, calculated from Eqs. (7) and (10). a) $\rho = 0.1$; b) 0.5; c) 0.9. The cavity depths L corresponding to the curves calculated from (10) are governed by the values of l at which the given curves are cut off (L = 0.5, 1, 2, 4, 6, 8, and 10, respectively). The dashed curves are calculated from Eq. (7) and the solid curves from Eq. (10).

Fig. 4. Emissivity of cylindrical cavities of various depths L as a function of the angle χ (degrees) from the cylinder axis. $\rho = 0.7$.

Substituting the value of ε_σ from (7) into this equation and using $\int_{(s)} u_\sigma dA_\sigma = A_0$ for the case of a flat aperture [1], we find

$$\varepsilon_h = 1 - \frac{1}{A_0} \int_{(s)} \frac{u_\sigma^2 \, dA_\sigma}{[\varepsilon/(1-\varepsilon)] + u_\sigma} . \tag{14}$$

Equation (14) appears to be different in form from the equation derived in [1] on the basis of similar assumptions,

$$\varepsilon_h = \frac{\varepsilon}{\varepsilon + [(1-\varepsilon)/A_0] \int_{(s)} u_\sigma^2 \, dA_\sigma} , \tag{15}$$

where the quantity $(1/A_0) \int_{(s)} u_\sigma^2 dA_\sigma$ is the average angular coefficient for the entire cavity, but we should not expect large numerical differences between the results found from the two equations.

Substituting the more accurate values of ε_σ given by (10) into (13), we find

$$\varepsilon_h = \varepsilon \left[1 + \frac{1}{\pi A_0} \iint_{(s)} \frac{u_s d\Omega_\sigma(s) \, dA_\sigma}{\varepsilon/(1-\varepsilon) + u_s} \right] . \tag{16}$$

Table 1 and Figs. 3 and 4 show the calculated emissivity of a cylindrical cavity having a flat bottom and a depth L, obtained for points on the bottom a distance r from the center of the bottom and for points of the cylindrical surface at a distance l from the open end of the cylinder (all dimensions are divided by the radius of the cylinder; i.e., this radius is set equal to 1). Table 1 shows the emissivity of the center of the bottom, $\varepsilon_{r=0}$, that for points on the bottom near the lateral surface, $\varepsilon_{r=1}$, the average emissivity of the bottom, $\bar{\varepsilon}_r$, and the value of ε_h as functions of L for the case $\varepsilon = \rho = 0.5$. We see that Eqs. (10) and (14) yield results which are very nearly equal to the data of [3], obtained by solving integral equation (9) on a computer by the method of successive approximations without any assumptions. Although the results found from Eq. (7) are slightly less accurate, they differ much less from the exact values than do the results calculated from the Gouffe [4] and Buckley [6] equations and are suitable for practical estimates of the emissivity. Figure 3 shows the emissivity of the cylindrical surface of cavities of various depths L as a function of l for three values of ρ. We see that the discrepancy between the results calculated from Eqs. (7) and (10) is greatest for points on the lateral surface near the bottom of the cylinder. This discrepancy decreases with decreasing ρ. Figure 4 shows the emissivity characteristics of cylindrical cavities in the case $\rho = 0.7$. Equation (7) was used to calculate these characteristics. We see that the emissivity increases primarily only in directions near the axis of the cylinder with increasing cavity depth L.

We turn now to some equations which can be used to calculate the angular coefficients of several of the cavities which are most commonly used in practice. These solutions were obtained on the basis of the theory of a radiation field [2]. The angular coefficient was found as the ratio of the illumination produced at a given point by the glowing surface of an aperture having a uniform luminosity to the magnitude of this luminosity. Multiplying the numerator and denominator in (8) by the brightness attributed to the aperture, we find

$$u_\sigma = \frac{B_0\Omega_\sigma(0)}{\pi B_0} = \frac{E_\sigma(0)}{R_0} = \frac{\mathscr{E}_\sigma(0)\cos\theta_0}{R_0}. \quad (17)$$

1. A circular cylinder open at one end; the notation and the results for this case are given above.

For points on the bottom of the cavity we have

$$u_r = \frac{1}{2}[1-(L^2-1+r^2)/\sqrt{(L^2-1+r^2)^2+4L^2}]. \quad (18)$$

For an axial element of the bottom of the cavity we have r = 0, so that we can write

$$u_{r=0} = (L^2+1)^{-1} \quad (19)$$

and

$$\varepsilon_{r=0} = \frac{\varepsilon(L^2+1)}{\varepsilon L^2+1}. \quad (20)$$

The average value of $\bar{u}_r$ for the entire bottom is

$$\bar{u}_r = 1-(L/2)(\sqrt{L^2+4}-L). \quad (21)$$

For points on the cylindrical surface,

$$u_l = \frac{1}{2}\left(\frac{l^2+2}{\sqrt{l^2+4}}-l\right) = \frac{2}{l(l^2+4)+(l^2+2)\sqrt{l^2+4}}. \quad (22)$$

If the cylinder is open at both ends (a circular aperture),

$$u_l = \frac{1}{2}\left[\frac{l^2+2}{\sqrt{l^2+4}}+\frac{(L-l)^2+2}{\sqrt{(L-l)^2+4}}-L\right]. \quad (23)$$

2. A rectangular recess of infinite length and a half-width of 1.

For the lateral faces of the recess, we have

$$u_l = \frac{1}{2}(1-l/\sqrt{l^2+4}), \quad (24)$$

where l is the distance from the edge of the recess into its interior along a face.

For the bottom of the recess we have

$$u_r = \frac{1}{2}\left[\frac{1+r}{\sqrt{(1+r)^2+L^2}}+\frac{1-r}{\sqrt{(1-r)^2+L^2}}\right]. \quad (25)$$

3. A truncated cone with an open smaller base, whose radius is set equal to 1:

$$u_l = \frac{\sqrt{4(1+l\sin\beta)+l^2}-2\sin\beta-l}{l\sqrt{4(1+l\sin\beta)+l^2}+4(1+l\sin\beta)+l^2}. \quad (26)$$

Here the distance l is measured from the aperture along the generatrix.

4. A cone with an open base of unit radius:

$$u_l = \frac{\sqrt{4L(L-l)+L^2l^2}+2-Ll}{l\sqrt{4L(L-l)+L^2l^2}+4(L-l)+Ll^2} \tag{27}$$

(l is the distance from the base to the vertex of the cone along a generatrix, whose length is L).

5. A wedge-shaped recess of infinite length, whose half-width is set equal to 1:

$$u_l = \frac{1}{2}\left[1 - \frac{Ll-2}{\sqrt{L^2(l^2+4)-4Ll}}\right], \tag{28}$$

where the distance l is measured from the rim of the recess to its edge along the face whose width is L.

6. A sphere with a circular aperture.

The angular coefficients (and thus the emissivity) are the same for all points of the sphere, given by

$$u = \sin^2\frac{\delta}{4} = \frac{1}{2}(1 \mp \sqrt{1-\gamma^2}). \tag{29}$$

Here the minus sign is used if the part of the spherical surface replaced by the aperture is smaller than a hemisphere ($\delta < \pi/2$), and the plus sign is used in the opposite case ($\delta > \pi/2$).

7. A cylindrical recess of infinite length (a circular tube with a slit along its entire length):

$$u_\alpha = \sin\frac{\delta}{4}\sin\frac{\alpha}{2} \quad \left(\alpha \geqslant \frac{\delta}{2}\right). \tag{30}$$

The average angular coefficient for the entire slit is, for the direction ψ,

$$u_\psi = (4/\delta)\sin^2\frac{\delta}{4}\cos\psi \quad \left(\psi \leqslant \frac{\pi-\delta}{2}\right). \tag{31}$$

Some of these equations can be used to determine the angular coefficients for cavities of several other shapes. For example, Eq. (18) also holds for points on the flat base (or bottom) of a truncated cone, regardless of its dimensions, and Eq. (25) can be used for points on the bottom of a trapezoidal recess.

We note in conclusion that by using the general equations and method for calculating the angular coefficients outlined above we can determine the emission characteristics of diffuse isothermal cavities of essentially any shape.

NOTATION

A_0	is the area of the aperture in the cavity;
A_S	is the surface area of the cavity walls;
B_0	is the brightness attributed to the surface of the cavity in the aperture;
$E_S(0)$	is the direct illumination at point s due to the glowing aperture surface;
$\mathscr{E}_S(0)$	is the magnitude of the light vector at point s;
k	is the index of the geometric progression in the series constructed from the sequence of angular coefficients;
L	is the parameter characterizing the cavity depth;
l	is the coordinate measured from the aperture, giving the position of the point under consideration on the lateral surface of the cavity;
r	is the coordinate giving the position of the point on the bottom (or base) of the cavity, measured from the center of the cavity (the symmetry axis or the center);
R_0	is the luminosity attributed to the cavity surface;
s	is an arbitrary point on a cavity wall;
$u_n(s)$	is the angular coefficient of the n-th reflection for surface-area element dA_S, which includes the point s;
$u_S = u_1(s)$	is the angular coefficient for single (or first) reflection, equal to the fraction of the radiative flux emitted from a cavity after its first reflection from element dA_S;
x, y, z	are the Cartesian coordinates whose x0y plane is in the aperture plane and whose 0z axis is perpendicular to this plane, directed into the interior of the cavity;
α	is the central angle, measured from the symmetry plane of a cylindrical recess (half the angle δ) along the side of the slit to the given point on the surface of the recess;

β	is the angle between the axis and the generatrix of a conical cavity;
γ	is the ratio of the aperture radius to the radius of a spherical cavity;
δ	is the angle at which diametrically opposite points of the cavity aperture (or the edge of a slit in a recess) are visible from the center of a spherical cavity (or from the axis of a cylindrical recess);
$\varepsilon = 1 - \rho$	is the emissivity of the material of the cavity walls;
ε_s	is the emissivity of a surface element dA_s;
$\varepsilon(\varphi, \chi)$	is the emissivity, averaged over the aperture surface, in the direction specified by the angles φ and χ;
ε_h	is the hemispherical emissivity of the cavity aperture;
θ	is the angle between the axis of the solid angle $d\omega_s(0)$ and the normal to the surface at point s;
θ_0	is the angle between the direction of the light vector, determined by the bright aperture, and the normal to the surface at point s;
ξ and η	are the coordinates of an arbitrary point in the plane of the aperture and within its boundary;
ρ	is the reflection coefficient of the material of the cavity walls;
σ	is the given point on the surface of the cavity walls;
φ	is the angle between the x0z plane and the plane passing through the 0z axis and the given observation direction;
χ	is the angle between the observation direction and the 0z axis;
ψ	is the dihedral angle with edge at the axis of a cylindrical recess, measured from the symmetry plane of the recess;
$d\omega_s(0)$	is the solid-angle element subtended by surface-area element dA_0 in the aperture at point s;
$d\Omega_s(0)$	is the projection of the solid-angle vector $d\omega_s(0)$ onto the normal to the surface at point s [2].

The symbols $\int\limits_{(0)}$ and $\int\limits_{(s)}$ denote integrations over the aperture surface and over the surfaces of the cavity walls, respectively; the right-hand subscripts on ω and Ω give the point at the vertex of the solid angle; and the subscripts in parentheses correspond to the surface element inscribed in the given solid angle.

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